Terry's

MATH TUTORING

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Livestream #1

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Grade 12 - Trigonometry

Converting Between Sums and Products of Trigonometric Functions

Goals

- Develop formulas to convert a sum or difference involving sines and cosines to a product of sines and cosines.
- Develop formulas to convert a product involving sines and cosines to a sum or difference of sines and cosines.
- Use these formulas to prove trigonometric identities.
- Use these formulas to solve trigonometric equations.

Notes

In these notes we will talk about how to convert sums of trigonometric functions into products and how to convert products of trigonometric functions into sums.

Let's take the two basic formulas for the sine of a sum and the sine of a difference as our starting point.

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

Adding these formulas gives:

$$\boxed{2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta)} \tag{1}$$

Subtracting these formulas gives:

$$\boxed{2\cos\alpha\sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta)} \tag{2}$$

These formulas are equivalent. To prove that the second formula is the same as the first, switch the roles of α and β in the second formula and use the fact that sin is an odd function.

Now take the two basic formulas for the cosine of a sum and the cosine of a difference as our starting point.

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

Adding these formulas gives:

$$\boxed{2\cos\alpha\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta)} \tag{3}$$

Subtracting these formulas gives:

$$\boxed{-2\sin\alpha\sin\beta = \cos(\alpha+\beta) - \cos(\alpha-\beta)} \tag{4}$$

The formulas $(1), (2), (3), (4)$ give us a way of passing from a product consisting of sine or cosine functions to a sum or difference of sine or cosine functions.

We will make a change of variables. Define

$$x = \alpha+\beta \qquad \text{and} \qquad y = \alpha-\beta$$

Solving for α and β gives

$$\alpha = \frac{x+y}{2} \qquad \text{and} \qquad \beta = \frac{x-y}{2}$$

Formulas (1) and (2) then respectively become:

$$\boxed{\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)} \tag{5}$$

$$\boxed{\sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)} \tag{6}$$

Similarly, formulas (3) and (4) then respectively become:

$$\boxed{\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)} \tag{7}$$

$$\boxed{\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)} \tag{8}$$

The formulas (5), (6), (7), (8) give us a way of passing from a sum or difference of sine or cosine functions to a product of sine or cosine functions.

Formula Sheet

Let's summarize in one place the formulas that we have derived. Dividing out the 2 in formulas (1), (2), (3), (4), gives the formulas needed to convert from a product of trigonometric functions to a sum.

$$\boxed{\sin\alpha\cos\beta = \frac{1}{2}(\sin(\alpha+\beta)+\sin(\alpha-\beta))} \tag{9}$$

$$\boxed{\cos\alpha\sin\beta = \frac{1}{2}(\sin(\alpha+\beta)-\sin(\alpha-\beta))} \tag{10}$$

$$\boxed{\cos\alpha\cos\beta = \frac{1}{2}(\cos(\alpha+\beta)+\cos(\alpha-\beta))} \tag{11}$$

$$\boxed{\sin\alpha\sin\beta = \frac{-1}{2}(\cos(\alpha+\beta)-\cos(\alpha-\beta))} \tag{12}$$

And we repeat the four formulas (5), (6), (7), (8) derived above for converting a sum or difference of trigonometric functions to a product.

$$\boxed{\sin x+\sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)} \tag{13}$$

$$\boxed{\sin x-\sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)} \tag{14}$$

$$\boxed{\cos x+\cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)} \tag{15}$$

$$\boxed{\cos x-\cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)} \tag{16}$$

These seven formulas will be the key to solving the following problems.

Problems

Easy

Write each trigonometric product as a sum.

1. $\sin 5x \cos 2x$
2. $2\cos 5t \sin t$
3. $\cos 2y \cos 8y$
4. $\sin\left(\frac{\pi}{2} - x\right)\sin\left(\frac{\pi}{2} + x\right)$
5. $\sin(u+v)\cos(u-v)$

Write each trigonometric sum as a product.

6. $\sin 3t + \sin t$
7. $\cos 6t - \cos 4t$
8. $\sin \dfrac{5\pi}{14} + \sin \dfrac{\pi}{14}$
9. $\cot 2 \cos 3 - \cos 7 \cot 2$

Intermediate

Find the general solution of the equation.

10. $\sin(5x) + \sin(3x) = \frac{3}{2}\cos x$

Prove the identities

11. $\sin \dfrac{2\pi}{9} + \sin \dfrac{\pi}{9} = \cos \dfrac{\pi}{18}$
12. $\dfrac{\cos 3t + \cos t}{\sin 3t - \sin t} = \cot t$
13. $\dfrac{\sin x - \sin y}{\cos x + \cos y} = \tan\left(\dfrac{x-y}{2}\right)$
14. $\dfrac{\cos t + \cos 3t}{2} = \dfrac{1 - 2\sin^2 t}{\sec t}$
15. $\dfrac{\sin(2x-y) + \sin y}{\cos(2x-y) + \cos y} = \tan x$

◆ Difficult

Prove the identities

16. $\dfrac{\sin\theta + \sin 3\theta + \sin 5\theta}{\cos\theta + \cos 3\theta + \cos 5\theta} = \tan 3\theta$

17. $\sin\theta + \sin 3\theta + \sin 5\theta + \sin 7\theta = 4\cos\theta\cos 2\theta\sin 4\theta$

18. $4\cos 3x\sin 2x\sin 5x = \cos 6x - \cos 4x - \cos 10x + 1$

Answers

1. $\frac{1}{2}\sin(7x) + \frac{1}{2}\sin(3x)$
2. $\sin 6t + \sin 4t$
3. $\frac{1}{2}\cos 10y + \frac{1}{2}\cos 6y$
4. $\frac{1}{2} + \frac{1}{2}\cos 2x$
5. $\frac{1}{2}\sin 2u + \frac{1}{2}\sin 2v$
6. $2\sin 2t \cos t$
7. $-2\sin 5t \sin t$
8. $2\sin \frac{3\pi}{14} \cos \frac{\pi}{7}$
9. $-2\cos 2 \sin 5$
10. $x = \pi/2 + k\pi$ or $x = \frac{1}{4}\sin^{-1}(3/4) + (\pi/2)k$ or $x = \frac{1}{4}(\pi - \sin^{-1}(3/4)) + (\pi/2)k$ where $k \in \mathbb{Z}$.